# Hausdorff Moment Problem and Associated Operator Theory

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**Abstract**

In this expository article, the association of some special classes of functions on semigroups and bounded operators on Hilbert spaces is expolered. In the process , we begin with the classical moment problems, make relevant historical comments and touch upon the recent developments in this area.

In this lecture, we propose to exhibit an amalgamation of Harmonic Analysis and Operator Theory. Some special type of functions get naturally tied up with some special type of operators on Hilbert spaces. This fruitful synthesis of two seemingly different areas has its roots in the classical moment problem.

The moment problem was first posed by Chebyshev in 1873 and then substantially took forward by Stieltjes, Markov, Hausdorff and Widder. In view of its connection with bounded operators on Hilbert spaces, we state here the Hausdorff Moment Problem. For the historical remarks and thorough discussion on moment problem, the reader is referred to[**?**], [**?**]

**Hausdorff Moment Problem**:
Given a sequence of non-negative real numbers $\{a_n\}$, does there exist a measure $\mu$ on $[0, 1]$ such that $a_n = \displaystyle\int_0^1 x^n d\mu$.

The following remarkable theorem was proved by Hausdorff [**?**] , which describes the solution of the one-dimensional moment problem on finite interval.

**Theorem 0.1.** *The following statements are equivalent:*
*(a) A sequence of non-negative real numbers $\{a_n\}$ is a moment sequence*

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*(b)* $\nabla^k a_n = \sum_{j=0}^{k}(-1)^j \binom{k}{j} a_{n+j} \geq 0, \ n \geq 0, k \geq 0$

The infinitely many conditions on $\{a_n\}$ in (b) allow one to define a measure $\mu$ on $[0,1]$ such that $a_n = \int_0^1 x^n d\mu$.
A new proof of Hausdorff's Theorem was later supplied by Hildebrandt and Schohenberg [**?**]. In fact they solved the multi-dimensional version of the problem. A sequence $\{a_n\}$ satisfying conditions in (b) is called a completely monotone sequence. This terminology was introduced by Widder [**?**].
It can be easily checked that the sequences $\{1\}$ and $\{\frac{1}{n+1}\}$ are completely monotone and hence moment sequences.
A function theoretic description of complete monotonicity was introduced much earlier by Berstein [**?**].

**Definition 0.2 :** A function $f : (0, \infty) \to \mathbb{R}$ is called completely monotone if $f$ is of class $C^\infty$ and $(-1)^n f^{(n)}(x) \geq 0, \ \forall n \geq 0, \ x > 0$.

**Example 0.3 :**

1. $f(x) = \frac{1}{x+1}$ is completely monotone

2. $f_t(x) = e^{xt}, t > 0$ is a family of completely monotone functions

The following theorem characterizes the class of completely monotone functions:

**Theorem 0.4.** *Let $f : (0, \infty) \to \mathbb{R}$ be a completely monotone function. Then there is a unique measure $\mu$ on $(0, \infty)$ such that $f(x) = \int_0^\infty e^{-xt} d\mu(t)$.*
*Conversely if $\int_0^\infty e^{-xt} d\mu(t) < \infty$, then $x \to \int_0^\infty e^{-xt} d\mu(t)$ is completely monotone.*

The integral representation of a completely monotone map is called the Laplace Transform of a measure $\mu$.
The above theorem at once generates the examples of completely monotone functions on $(0, \infty)$. Naturally, the restriction of a completely monotone function to the set of non-negative integers is a completely monotone sequences and thus a moment sequences.
At this stage we point out that the set of completely monotone maps forms a subclass of the class of positive definite functions on the semi-group $\mathbb{R}_+$ of non-negative real numbers. For the theory of

positive definite functions as well as for the historical comments on this area, the reader is referred to the excellent book [**?**]. Some important properties of completely monotone functions are listed below:

**Proposition 0.5.** *1. The set of completely monotone functions is a convex cone i.e.* $sf_1 + tf_2 \in \mathcal{CM}$ *for all* $s, t \geq 0$ *and* $f_1, f_2 \in \mathcal{CM}$.

*2. The class* $\mathcal{CM}$ *is closed under multiplication i.e.* $f_1 f_2 \in \mathcal{CM}$ *for* $f_1, f_2 \in \mathcal{CM}$.

*3. The class* $\mathcal{CM}$ *is closed under pointwise convergence i.e.* $f_n \to f$ *and* $f_n \in \mathcal{CM}$ *then* $f \in \mathcal{CM}$.

*4.* $(-1)^n \Delta_{a_n} \cdots \Delta_{a_2} \Delta_{a_1} f \geq 0$, *for all* $n$ *and for any* $a_1, a_2, \cdots a_n \in \mathbb{R}_+$

The function theory developed in the beginning of the twentieth century as described above hinges around the moment problem and related classes of positive definite functions on the semi-groups on $\mathbb{R}_+$ and $\mathbb{N}$.
We now introduce a subclass of the class of negative definite functions on $\mathbb{R}_+$.
Definition: A function $f : (0, \infty) \to \mathbb{R}$ is called a Bernstein function if $f$ is of class $C^\infty$ and $(-1)^{n-1} f^{(n)}(x) \geq 0, \ \forall n \in \mathbb{N}$.
The Bernstein functions were extensively studied by Schohenberg [**?**], Bochner[**?**]. In particular the following remark relates this class to the class of completely monotone functions.

**Remark 0.6 :** A non-negative $C^\infty$ function on $(0, \infty)$ is Bernstein function if and only if $f'$ is completely monotone function.
The above observation coupled with the integral representation of completely monotone maps result into the integral representation of Bernstein functions. This representation is known as Levy-Khincin representation.

**Theorem 0.7.** *A function* $f : (0, \infty) \to \mathbb{R}$ *is a Bernstein function if and only if it admits the representation*
$f(x) = a + bx + \int_{(0,\infty)} (1 - e^{-xt}) d\mu(t)$, *where* $a, b \geq 0$ *and* $\mu$ *is a measure on* $(0, \infty)$ *satisfying* $\int_{(0,\infty)} (1 \wedge t) d\mu(t) < \infty$.

By choosing appropriate measures on $(0, \infty)$ one can check that $x^\alpha$ $(0 < \alpha < 1)$, $\frac{x}{1+x}$, $\log(1+x)$ are Bernstein functions.

The class of sequences associated with the Bernstein functions are the completely alternating sequences. A detailed discussion on completely alternating sequences can be found in [?].

**Definition 0.8 :** A sequence $\{a_n\}$ of non-negative real numbers is called completely alternating if $\nabla^k a_n = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a_{n+j} \leq 0, \ n \geq 0, k \geq 1.$

We now turn our attention to a significant result by Sz. Nagy which connects the moment problem with the theory of bounded linear operators on Hilbert spaces.

**Theorem 0.9.** *A sequence of operators $M_n$ satisfies* $\sum_{k=0}^{n} \binom{n}{k} M_k M_{n-k} \geq 0$ *for every $n$ if and only if there exists an operator measure $A$ on $[0,1]$ such that* $M_n = \int_0^1 t^n dA.$

Sz. Nagy's paper on the operator valued version of the Hausdorff moment theorem appeared in 1952[?] . In the same period, Halmos introduced the class of subnormal operators. We shall define subnormal operators for the ready reference.

**Definition 0.10 :** Definition: An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be subnormal if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and a normal operator $N \in \mathcal{B}(\mathcal{K})$ such that $N\mathcal{H} \subset \mathcal{H}$ and $N|\mathcal{H} = S$.

A detailed discussion of subnormality including various characterizations of the notion can be found in a scholarly work by Conway [?]. We shall now quote Embry's characterization of subnormality [?] which is relevant to the present discussion.

**Theorem 0.11.** *An operator $T$ is subnormal if and only if there exists an operator measure $A$ concentrated on an interval $[0,a]$ such that* $T^{*n}T^n = \int_0^a t^{2n} dA$ *for all $n \geq 0$.*

Jim Agler [?] aptly coupled the results of Sz. Nagy and Embry and came up with a brilliant characterization of subnormal contractions:

**Theorem 0.12.** *An operator $T$ on a Hilbert space $\mathcal{H}$ is a subnormal contraction if and only if* $\sum_{k=0}^{n} \binom{n}{k} T^{*k}T^k \geq 0$ *for all $k \geq 0, n \geq 0$.*

Agler also constructed a sequence of subnormal contractions. The unilateral shift and Bergman shift are the prominent among these and have been studied extensively.
Athavale and Pedersen [?] generalized Agler's work to the multi-dimensional set up. Agler's criterion allows one to relate complete monotonicity with subnormality in the following sense:
An operator $T$ on a Hilbert space $\mathcal{H}$ is a subnormal contraction if and only if the sequence $\phi_h(n) = ||T^n h||^2$ is completely monotone for every $h$. Specializing to weighted shift operators, we have:
A weighted shift operator $T$ with weight sequence $\alpha_n$ is subnormal if and only if the sequence $\beta_n = \alpha_0^2 \cdots \alpha_{n-1}^2$ is completely monotone.
Subnormal contractions, thus get naturally tied up with the class of completely monotone sequences which is a subclass of positive definite functions on the semigroup $\mathbb{N}$ of non-negative integers.
We shall now discuss the class of operators antithetical to the class of subnormal contractions and get naturally associated with completely alternating sequences which is a subclass of negative definite functions on the semigroup $\mathbb{N}$ of non-negative integers. Such type of operators were introduced and studied in the works of Aleman [?] and Richter[?]. A major impetus in this direction was given by Athavale [?] . In this paper he introduced a class of Completely Hyperexpansive Operators. As can be observed from the definition below, the hyperexpansive operator is defined through infinitely many negative conditions by reverting the inequalities in Agler's criterion.

**Definition 0.13 :** An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be completely hyperexpansive if $\sum_{k=0}^{n} (-1)^k \binom{n}{k} T^{*k} T^k \leq 0$, for all n $\geq 1$.
Along with the isometries, an important example of a completely hyperexpansive operator is the Dirichlet shift; the weighted shift operator with weight sequence $\{\frac{n+2}{n+1}\}$.

It is evident from the definition that an operator $T$ on a Hilbert space $\mathcal{H}$ is completely hyperexpansive if and only if the sequence $\phi_h(n) = ||T^n h||^2$ is completely alternating for every $h$.
Athavale explored the intrinsic relationship between completely monotone map and Bernstein function and translated it into corresponding relation between subnormal contractions and completely hyperexpansive operators.
In particular an operator theoretic manifestation of the following important result leads to the computation of the Cauchy dual of a completely hyperexpansive weighted shift:
Theorem: Let $\psi : (0, \infty) \to \mathbb{R}$. Then $\psi \in \mathcal{BF}$ if and only if $e^{-t\psi} \in \mathcal{CM}$.
The structural and spectral properties of completely hyperexpansive operators were further explored in [?] ,[?]. S. Chavan took the subject further by proving several interesting results on completely hyperexpansive operators and related classes in one and several dimensions [?], [?], [?].

The theme of relating certain special class of functions to a class of operators is still flourishing. In the recent work by Chavan and Sholapurkar [**?**], [**?**] a class of completely monotone functions of finite order which encompasses polynomials as well as a class of Bernstein functions of finite order which contains polynomials have been discussed. The operator theoretic manifestations of these classes provide interesting new examples of operators and also help in supplying new proofs of known results.
The subtle interconnection between various classes of functions on semigroup and exploring their reflections in the interactions between corresponding classes of operators is a rewarding experience.

## References


[1] J. Agler , *Hypercontractions and Subnormality*, J. Operator Theory **13** (1985), 203-217.

[2] N. Akhiezer, The classical Moment Problem , Oliver and Boyd, London, 1961.

[3] A. Aleman, *The multiplication operators on Hilbert spaces of analytic functions*, Habilitation-sschrift, Fernunuversitat Hagen, 1993.

[4] A. Athavale, *On completely hyperexpansive operators*, Proc. Amer. Math. Soc. **124** (1996), 3745-3752.

[5] A. Athavale and S. Pedersen, *Moment problems and subnormality*, J. Math. Anal. Appl. **146** (1990), 434-441.

[6] A. Athavale and V. Sholapurkar, *Completely hyperexpansive operator tuples*, Positivity, **3** (1999), 245-257.

[7] C. Berg, J. P. R. Christensen, and P. Ressel, *Harmonic Analysis on Semigroups*,

[8] S. Bernstein , *Sur la definition et les proprietes des fonctions analytiques d'une variable reele* , Math. Ann. **75** (1914), 449-468.

[9] S. Bochner , *harmonic Analysis and the Theory of Probability*, California University Press, Berkeley, 1955 .

[10] S. Chavan, *On operators Cauchy dual to 2-hyperexpansive operators*, Proc. Edin. Math. Soc. **50** (2007), 637-652.

[11] S. Chavan, *On operators close to isometries*, Studia Math. **186** (2008), 275-293.

[12] S. Chavan, *$C^*$-algebras generated by spherical hyperexpansions*, preprint.

[13] S. Chavan, *Essential normality of operators close to isometries*, preprint.

[14] S. Chavan and R. Curto, *On Operators Cauchy Dual to 2-hyperexpansive Operators: The Multivariable case*, preprint.

[15] S. Chavan and V. Sholapurkar, *Completely Monotone Functions of Finite Order and Agler's Conditions*, Studia Math, **226** (2015), 229-258.

[16] S. Chavan and V. Sholapurkar, *Completely Hyperexpansive Tuples of finite order*, Preprint .

[17] J. Conway, *The Theory of Subnormal Operators*, Math. Surveys Monographs, vol 36, Amer. Math. Soc. Providence, RI 1991.

[18] . Embry , *A generalization of the Halmos-Bram criterion for subnormality* , Acta. Sci. Math. (Szeged) **35** (1973), 61-64.

[19] F. Hausdorff, *Momentenprobleme fur ein endliches Intervalla* , Math Z. , **14** (1923), 220-248.

[20] T. Hildebrandt and I. Schoenberg *On linear functional operators and the moment problem for the finite interval in one or several dimensiona* , Ann. of Math. **34** (1933), 317-328.

[21] Z. Jabłoński, *Complete hyperexpansivity, subnormality and inverted boundedness conditions*, Proc. Edin. Math. Soc. **44** (2001), 613-629.

[22] S. Richter, *Invariant subspaces of the Dirichlet shift*, J. Reine Angew. Math. **386** (1988), 205-220.

[23] I. Schohenberg , *On absolutely convex functions(abstarct)*, Bull. Am. Math. Soc. **47** (1941), 389.

[24] V. Sholapurkar and A. Athavale, *Completely and alternatingly hyperexpansive operators*, Jour. Op. Th. **43** (2000), 43-68.

[25] Sohat and Tamarkin, *The problem of moments*, Amer. Math. Soc. New York, 1943

[26] B. Sz. Nagy, *A moment problem for self adjoint operators*, Acta Math. Acad. Sci. Hungar. **3** (1952), 285-292.

[27] D. Widder, *Necessary and sufficient conditions for the representation of a functiona as a Laplace integral* , Trans. Am. Math. Soc. **33** (1931), 851-892.